

## NICE DECISION SCHEMES\*\*

### 1. INTRODUCTION

This paper deals with a particular type of social decision-making procedures, which I call 'decision schemes'. In general, a social decision-making procedure is a rule which to each specification of the preferences of the individuals in a society with regard to the alternatives open to this society assigns a criterion to choose among such alternatives. Different types of decision-making procedures may arise, depending on:

- (1) the nature of individual preferences over alternatives; and
- (2) the kind of choice criterion provided by the procedure.

Decision schemes are defined so that they specify a lottery over the set of alternatives for each  $N$ -tuple of rankings over alternatives—where a lottery is a probability measure and a ranking is a complete, transitive and asymmetric binary relation. They can thus be interpreted as rules which process the ordinal features of individual preferences and establish the probability with which each of the alternatives is to be chosen on the basis of such preferences.

Traditional social choice theory has been generally concerned with decision-making procedures of a different nature, which I will call deterministic because they assign a definite choice of alternatives to each specification of individual preferences. This is the case, for example, of Arrow's Social Welfare Functions, Sen's Collective Choice Rules and Fishburn's Social Choice Functions. Actually, even within these structures there is some room left for chance: whenever more than one alternative is chosen by the procedure, one can interpret that the final decision among the chosen alternatives will be left to chance. On the other hand, some decision schemes may well be deterministic in a certain sense. Think, for example, of those which always assign trivial lotteries where an alternative has probability one of being chosen. Yet, the structure of decision schemes is such that the

probability of some alternative being chosen plays an explicit role, while probabilistic considerations are at most implicit in what I choose to call deterministic procedures. In this restricted sense the distinction is, I think, a valid one.

Decision schemes are but one possible type of social decision-making procedures which explicitly rely on chance. Different procedures can be specified by changing the domain and/or the range over which they operate. Thus, for example, Zeckhauser [6] has studied the class of voting schemes, which allow for cardinal preferences of individuals in their range, and include decision schemes as a particular case. Intriligator [5] considered probabilistic choice rules defined over vectors to be interpreted as the preferences of uncertain voters. There is a growing interest for the study of probabilistic decision-making procedures in general. In particular, decision schemes seem to be important because they are compatible with the traditional view in economics that only the ordinal features of individual preferences are to count in the process of choice. Decision schemes have been studied by Gibbard [4], Zeckhauser [6] and Fishburn. They also encompass, as a particular case, the lottery rules over sets of two alternatives considered by Fishburn and Gehrlein [3].

The purpose of this paper is threefold: (a) to define some classes of decision schemes which constitute natural adaptations of well-known deterministic procedures to the probabilistic setting; (b) to identify a number of conditions on decision schemes which appear to be attractive; and (c) to characterize the class of decision schemes which satisfy these conditions. Objectives (b) and (c) correspond to what Fishburn [2] has called, respectively, the 'analytic' and the 'synthetic' approaches in social choice theory. As for the description of specific procedures (objective (a)) it has a long tradition in social choice theory, as it is from the study of specific proposals that more general questions do generally arise.

I define two classes of decision schemes. *Supporting size schemes* assign a certain weight to the size of the group of individuals which prefer an alternative  $x$  over another alternative  $y$ . The probability that they attach to each alternative is the sum of the weights which correspond to the alternative on the basis of its supporting sizes in pairwise comparisons with all others. *Point voting schemes* assign a certain weight to each position in an individual ranking. The probability that they attach to each alternative is the sum of the weights which

correspond to the alternative on the basis of its position in the ranking of each of the individuals.

Supporting size decision schemes can be viewed as an extension of the system of majority voting. The random character of their outcome allows for them to encompass simultaneously the spirit of simple and qualified majority voting. Point voting decision schemes can be viewed as an extension of deterministic procedures based on scoring, of which Borda's count is the best known example. While the principles of scoring and majority voting are in general mutually exclusive as the basis for deterministic choices, they are partially compatible when social outcomes are lotteries.

Also, a number of conditions on decision schemes are put forward which extend notions from the theory of deterministic social choice. A scheme is called *anonymous* if permutating the social order of individuals without changing their preference rankings does not change the probabilities attached to alternatives. A scheme is *neutral* if any permutation of the alternatives in the rankings of all individuals results in the corresponding permutation of the probabilities attached to each of the alternatives. A scheme is *strategy-proof* if no expected utility maximizer would ever find it to his advantage to misrepresent his preference ranking of certain prospects.

Finally, the connection between the above conditions and the proposed classes of decision schemes is established by means of the following theorem:

*A decision scheme is anonymous, neutral and strategy-proof if and only if it is a probability mixture of a point voting and a supporting size decision scheme.*

It has been known for some time that strategy-proofness, a very rare property for deterministic social decision-making procedures, can be satisfied when chance is allowed to play a role in selecting the social outcome. There is some argument as to how bad it is for a procedure not to be strategy-proof. But those that meet this requirement are certainly interesting, since they have the property that no individual will find it advantageous to engage in strategic considerations that might lead him to misrepresent his preferences. If chance helps in achieving strategy-proofness, this might constitute an argument in favour of its playing a role in social decision processes.

However, this argument would be a very weak one indeed if all strategy-proof decision schemes turned out to be undesirable by the standards which are generally used in evaluating social decision procedures, such as anonymity, neutrality or efficiency. One might ask the question, then, whether there exist 'nice' decision schemes which are strategy-proof. In this context, the theorem presented here could be interpreted as providing a quite positive answer: not only do there exist strategy-proof decision schemes which are anonymous and neutral, but they are, in fact, simple to operate mechanisms, based on rules that are easy to understand and which constitute natural extensions of the best known deterministic procedures. Therefore, strategy-proofness does not preclude the equal treatment of individuals and alternatives, and no overly complicated procedures are needed to achieve it. Unfortunately, however, the picture is not so satisfactory from the point of view of efficiency. Very natural efficiency requirements are violated by most of the decision schemes characterized above. But then, finding out about trade-offs of this kind is one of the main objectives of social choice theory.

The paper is organized as follows. After some preliminary definitions (Section 2), the classes of point voting and supporting size schemes are defined in Section 3. Section 4 presents a number of conditions on decision schemes. In Section 5 the theorem characterizing neutral, anonymous and strategy-proof decision schemes is provided. Section 6 presents some fundamental results, due to A. Gibbard, which are essential along the paper. Section 7, on the basis of these results, contains the actual proof of the theorem. Section 8 discusses some additional properties of the classes under study. Conclusions follow in Section 9.

## 2. NOTATION AND DEFINITIONS<sup>1</sup>

Let  $V$  be a finite set, called the set of *alternatives*. Elements of  $V$  are denoted by  $x, y, z, w, \dots$ .  $M$  denotes the cardinality of  $V$ .

A *ranking* of  $V$  is a binary relation  $P$  which, for all  $x$  and  $y$ , satisfies:

- (a) Connectedness:  $x \neq y \rightarrow (xPy \vee yPx)$
- (b) Asymmetry:  $xPy \rightarrow \sim yPx$ , and
- (c) Transitivity:  $(xPy \ \& \ yPz) \rightarrow xPz$ .

Let  $I = \{1, 2, \dots, N\}$  be an initial segment of the positive integers.  $I$  is called the *set of individuals*, and  $N$  is the *number of individuals*.

A *ranking  $N$ -tuple* over  $V$  is an  $N$ -tuple  $(P_1, \dots, P_N)$  of rankings of  $V$ . Ranking  $N$ -tuples are denoted by  $\mathbf{P}, \mathbf{P}', \mathbf{P}^*, \dots$ . Ranking  $N$ -tuples are interpreted as functions which assign to each individual  $i \in I$  the ranking  $P_i$  of  $V$ .

A *measure* over  $V$  is a function  $m$  which assigns a non-negative number  $m(x)$  to each member  $x$  of  $V$ . The sum  $\sum_{x \in V} m(x)$  is called the *weight* of the measure. A *lottery* is a measure of weight one. Lotteries will be denoted by  $l$ .

A *scheme* for  $\langle I, V \rangle$  is a function from the set of ranking  $N$ -tuples over  $V$  to the set of measures over  $V$  which have a fixed weight  $\alpha$ . The value  $\alpha$  is called the *weight of the scheme*. Schemes will be denoted by  $f, f', \dots$ .

A *decision scheme* is a scheme of weight one. Decision schemes will be denoted by  $d, d', \dots$ .

A decision scheme is, thus, a function which determines the probability with which each of the alternatives in  $V$  is to be selected, given the preferences of individuals expressed by a ranking  $N$ -tuple.

### 3. SOME PROPERTIES OF SCHEMES

Three conditions are now defined on schemes which adapt to the probabilistic framework the notions of anonymity, neutrality and strategy-proofness.

A *permutation* is a one-to-one function from a finite set onto itself.

Given a permutation  $\rho$  on the set of alternatives and a preference ranking  $P$ ,  $P^\rho$  is defined so that, for all  $x, y \in V$ ,  $xPy \leftrightarrow \rho(x)P^\rho\rho(y)$ .

Given  $\rho$  and an  $N$ -tuple of rankings  $\mathbf{P} = (P_1, \dots, P_N)$ ,  $\mathbf{P}^\rho$  is defined to be the  $N$ -tuple  $(P_1^\rho, \dots, P_N^\rho)$ .

Given a permutation  $\sigma$  on the set of individuals and an  $N$ -tuple of individual rankings  $\mathbf{P} = (P_1, \dots, P_N)$ ,  $\mathbf{P}^\sigma$  is defined to be the  $N$ -tuple of rankings  $\mathbf{P}^\sigma = (P_{\sigma^{-1}(1)}, \dots, P_{\sigma^{-1}(N)})$ .

A scheme  $f$  is *anonymous* iff, given any permutation  $\sigma$  on the set of individuals,  $f(x, \mathbf{P}) = f(x, \mathbf{P}^\sigma)$  for all  $x \in V$  and every  $N$ -tuple of rankings  $\mathbf{P}$ .

A scheme  $f$  is *neutral* iff, given any permutation  $\rho$  on the set of alternatives,  $f(x, \mathbf{P}) = f(\rho(x), \mathbf{P}^\rho)$  for all  $x \in V$  and every  $N$ -tuple of rankings  $\mathbf{P}$ .

Thus, an anonymous scheme is one which does not discriminate among individuals, and a neutral scheme is one that does not discriminate among alternatives.

$\mathbf{P}$  and  $\mathbf{P}'$  agree off  $k$  iff  $(\forall i)[(i \neq k) \rightarrow P_i = P'_i]$ .  $\mathbf{P}/_k \mathbf{P}$  is the ranking  $N$ -tuple  $\mathbf{P}'$  such that  $P'_k = P$  and  $\mathbf{P}'$  agrees with  $\mathbf{P}$  off  $k$ .

A utility scale  $U$  over  $V$  is an assignment of real numbers to the members of  $V$ . Where  $U$  is a utility scale over  $V$  and  $l$  is a lottery over  $V$ , the expected utility  $U(l)$  of  $l$  on scale  $U$  is defined by

$$U(l) = \sum_{x \in V} U(x)l(x).$$

Utility scale  $U$  fits a ranking  $P$  iff, for all  $x, y \in V$ ,

$$U(x) > U(y) \leftrightarrow xPy.$$

In the remainder of the section, conditions are defined which apply to decision schemes only.

A decision scheme  $d$  is *potentially manipulable* by  $k$  at  $\mathbf{P}$  if there is a utility scale  $U$  which fits  $P_k$  and a ranking  $P'_k$  of  $V$  such that, where  $\mathbf{P}' = \mathbf{P}/_k P'_k$ ,  $U(d\mathbf{P}') > U(d\mathbf{P})$ .

$d$  is *manipulable* if there are an individual  $k$  and a ranking  $N$ -tuple  $\mathbf{P}$  such that  $d$  is potentially manipulable by  $k$  at  $\mathbf{P}$ . Otherwise,  $d$  is *strategy-proof*.

If  $d$  is potentially manipulable and individual  $k$  is endowed with utility scale  $U$ , he has an incentive to find out about the preferences that other individuals will declare and, eventually, to misrepresent his preferences when by doing so he is able to change the social ranking  $N$ -tuple from  $\mathbf{P}$  to  $\mathbf{P}/_k P'_k$ .

If, on the contrary,  $d$  is strategy-proof, then no agent can ever find it advantageous to misrepresent his preferences, and there are no incentives for any single individual to engage in strategic considerations.

#### 4. TWO CLASSES OF SCHEMES

Two specific classes of schemes are now defined, which can be viewed as the probabilistic counterpart of well known types of deterministic social choice procedures.

The *rank* of an alternative  $x$  for an individual  $i$  within a ranking  $N$ -tuple  $\mathbf{P}$  is defined as  $r(i, x, \mathbf{P}) = |\{y \mid y \in V \ \& \ y P_i x\}| + 1$ . Clearly,  $1 \leq r(i, x, \mathbf{P}) \leq M$ , for all  $x, i$  and  $\mathbf{P}$ .

The *supporting size* for an alternative  $x$  over an alternative  $y$  within a ranking  $N$ -tuple  $\mathbf{P}$  is defined as

$$s(x, y, \mathbf{P}) = |\{i \mid i \in I \ \& \ x P_i y\}|.$$

Clearly,  $0 \leq s(x, y, \mathbf{P}) \leq N$  and  $S(x, y, \mathbf{P}) + s(y, x, \mathbf{P}) = N$  for all  $x, y$  and  $\mathbf{P}$ . A scheme  $f$  for  $(I, V)$  is a *point voting scheme* iff there exists an  $M$ -dimensional vector of real numbers  $(a_1, \dots, a_M)$ , to be called a *positional scoring vector*, such that:

- (1)  $a_1 \geq a_2 \geq \dots \geq a_M \geq 0$ , and
- (2) for all  $\mathbf{P}$  and  $x$ ,  $f(x, \mathbf{P}) = \sum_{i=1}^M a_{r(i,x,\mathbf{P})}$ .

It will be said, in this case, that  $f$  is representable as a point voting scheme with positional scoring vector  $(a_1, \dots, a_M)$ . Clearly, if  $d$  is a point voting decision scheme, it must be that

$$\sum_{i=1}^M a_i = \frac{1}{N}.$$

Point voting schemes operate in the following way: each alternative is given a score  $a_i$  every time an individual ranks it in  $i$ th position. The total weight assigned to an alternative for a given ranking  $N$ -tuple is the sum of the scores that it has obtained on the basis of the preferences of each individual. In the case of point voting *decision* schemes the positional scoring vector is chosen in such a way that the weights assigned to the different alternatives be probability distributions over  $V$ .

A scheme  $f$  for  $(I, V)$  is a *supporting size scheme* if there exists an  $N+1$ -dimensional vector of real numbers  $(b_N, \dots, b_0)$ , to be called the *supporting size scoring vector*, such that:

- (1)  $b_N \geq b_{N-1} \geq \dots \geq b_0 \geq 0$
- (2)  $\exists K$  such that  $b_j + b_{N-j} = K$  for all  $j \leq N/2$ , and
- (3) for all  $\mathbf{P}$  and  $x$   $f(x, \mathbf{P}) = \sum_{z \in V-x} b_{s(x,z,\mathbf{P})}$ .

We say then that  $f$  is representable as a supporting size scheme with supporting size scoring vector  $(b_N, \dots, b_0)$ . Clearly, if  $d$  is a supporting size decision scheme it must be that, for  $j \leq N/2$ ,  $b_j + b_{N-j} = 2/M(M-1)$ .

Supporting size schemes operate on the basis of pairwise comparisons among alternatives. If  $j$  individuals prefer  $x$  to  $y$ , then  $x$  is assigned score  $b_j$ . The total weight assigned to an alternative is the sum of the scores it has obtained on the basis of pairwise comparisons with all others. In the case of supporting size *decision* schemes, the supporting size scoring vector is chosen in such a way that the weights assigned to the different alternatives be probability distributions over  $V$ .

Point voting schemes constitute a natural probabilistic counterpart of a well-known type of deterministic rules for social choice, variously called scoring functions, representable voting functions, etc. . . . , and of which Borda's count is a classical example.

Supporting size schemes, on the other hand, convey the spirit of majority voting into the probabilistic framework. However, since decision schemes are more versatile than their deterministic counterparts, supporting size schemes are able to encompass, simultaneously, the features of simple and qualified majority voting.

Within a deterministic setting, the principles of majority voting and positional scoring are incompatible when the number of alternatives is greater than two. This is not the case within the framework of decision schemes, since there may exist decision schemes which are representable both as point voting and supporting size schemes. For example, where  $M=3$  and  $N=4$ , the supporting size decision scheme with supporting size scoring vector  $(5/18, 4/18, 3/18, 2/18, 1/18)$  can also be represented as a point voting scheme with positional scoring vector  $(5/36, 3/36, 1/36)$ .

##### 5. NEUTRAL, ANONYMOUS AND STRATEGY-PROOF DECISION SCHEMES

A scheme  $f$  is a *probability mixture* of schemes  $f_1, \dots, f_m$  iff there is a sequence  $\alpha_1, \dots, \alpha_m$ , where  $\alpha_i \geq 0$  for all  $i \in \{1, \dots, m\}$  such that, for every  $\mathbf{P}$  and  $x$ ,

$$f(x, \mathbf{P}) = \alpha_1 f_1(x, \mathbf{P}) + \dots + \alpha_m f_m(x, \mathbf{P}).$$

Where  $d$  is such a probability mixture, I will write

$$f = \alpha_1 f_1 + \dots + \alpha_m f_m = \sum_{i=1}^m \alpha_i f_i.$$

By definition,  $f'' = f - f'$  iff  $f = f' + f''$ .

Clearly, if a decision scheme  $d$  is the probability mixture of decision schemes  $d_1, \dots, d_m$ , it has to be that  $\sum_{i=1}^m \alpha_i = 1$ .

The main result of the paper can now be stated.

**THEOREM 1.** A decision scheme is anonymous, neutral and strategy-proof iff it is a probability mixture of a point-voting and a supporting size decision scheme.

The proof of this theorem relies very heavily on the work of Gibbard in [4]. In this remarkable paper, Gibbard provides a full characterization of strategy-proof decision schemes.

The class I characterize is therefore a subset of Gibbard's set. It is hoped that the characterization provided by Theorem 1 above will contribute (1) to clarify the specific contents of the class of strategy-proof decision schemes, (2) to point out, in the language of the theory, the possibility of locating some attractive features within this class, and (3) to establish a connection between these procedures and the probabilistic versions of such widely used collective decision-making procedures as those of majority voting and positional scoring. This result should also shed some light on the meaning of strategy-proofness and on the basic differences between the deterministic and the probabilistic setting with regard to this particular feature of collective decision-making procedures.

## 6. GIBBARD'S THEOREM ON STRATEGY-PROOF DECISION SCHEMES

Before proceeding to the proof of Theorem 1, some more definitions and two versions of Gibbard's Theorem need to be stated.

Scheme  $f$  is *unilateral* iff there is a  $k \in I$  such that, for all  $\mathbf{P}$  and  $\mathbf{P}'$ , if  $P_k = P'_k$ , then  $d\mathbf{P} = d\mathbf{P}'$ .

Scheme  $f$  is *duple* iff there are alternatives  $x$  and  $y$  such that for every other alternative  $z$  and all  $\mathbf{P}$ ,  $f(z, \mathbf{P}) = 0$ .

A subset  $X \subseteq V$  heads ranking  $P_k$  iff for any pair  $x, y \in V$ ,  $[x \in X \ \& \ y \notin X] \rightarrow xP_k y$ .

A scheme  $f$  is *localized* iff for every  $k \in I$ , for every  $\mathbf{P}$  and  $\mathbf{P}'$  which agree off  $k$  and for every set  $X$  that heads both  $P_k$  and  $P'_k$ ,  $f(X, \mathbf{P}) = f(X, \mathbf{P}')$ .

A *switch* is a reversal of two adjacent alternatives in a ranking.  $xP_k! y$  will mean that  $x$  immediately precedes  $y$  in ranking  $P_k$ . Where  $xP_k! y$ ,  $P_k^{yx}$  is the ranking which switches  $x$  and  $y$  in  $P_k$  and permutes no other alternatives.

Scheme  $f$  is *non-perverse* iff for every  $\mathbf{P}$ ,  $k$ ,  $x$  and  $y$  such that  $xP_k! y$ , where  $\mathbf{P}' = \mathbf{P}/_k P_k^{yx}$ ,  $f(y, \mathbf{P}') \geq f(y, \mathbf{P})$ .

Gibbard's Theorem characterizing the set of strategy-proof decision schemes can now be stated.

**THEOREM 2 (Gibbard).** A decision scheme  $d$  is strategy-proof iff it is a probability mixture of decision schemes, each of which is localized, non-perverse and either unilateral or duple.

The necessity part of the theorem can be given a more explicit formulation, which is going to be of use to prove Theorem 1. Before this formulation can be provided, some additional notions have to be introduced.

$\Gamma_x P_x$  is  $P_k$  with  $x$  moved to the bottom or grounded. Thus,

$$z \neq x \rightarrow z \Gamma_x P_k x \quad \text{for all } z \in V$$

$$x \notin \{y, z\} \rightarrow (y \Gamma_x P_k z \leftrightarrow y P_k z) \quad \text{for all } y, z \in V.$$

By definition,  $\Gamma_x \mathbf{P} = (\Gamma_x P_1, \dots, \Gamma_x P_N)$ .

Scheme  $f$  is *constant* iff for all  $x \in V$  and all  $\mathbf{P}$  and  $\mathbf{P}'$ ,  $f(x, \mathbf{P}) = f(x, \mathbf{P}')$ .

$k$ 's *unilateral component* of scheme  $f$  is the function  $f_k$  such that, for all  $x$  and  $\mathbf{P}$ ,  $f_k(x, \mathbf{P})$  is the minimal value, from among all  $\mathbf{P}'$  such that  $P_k = P'_k$ , of  $f(x, \mathbf{P}') - f(x, \mathbf{P}'/_k \Gamma_x P_k)$ .

The *xy duple component* of a scheme  $f$  is the scheme  $f_{xy}$  such that, for all  $\mathbf{P}$ ,  $f_{xy}(x, \mathbf{P}) = f(x, \Gamma_{xy} \mathbf{P})$ ,  $f_{xy}(y, \mathbf{P}) = f(y, \Gamma_{xy} \mathbf{P})$ , and for all  $z$  other than  $x$  and  $y$ ,  $f_{xy}(z, \mathbf{P}) = 0$ .

Now we can formulate more precisely the necessity part of Gibbard's theorem.

**THEOREM 2'** (Gibbard). If  $d$  is a strategy-proof decision scheme, it can be expressed as the sum of  $N + M(M-1)/2 + 1$  schemes, in the form

$$d = f_0 + \sum_{i=1}^N f_i + \sum_{x \in V} \sum_{\substack{z \neq x \\ z \in V}} f_{xz}, \quad \text{where}$$

- (1)  $f_0$  is a constant scheme such that, for all  $z \in V$ ,  $f(z, \mathbf{P}) = d(z, \Gamma_x \mathbf{P})$  for any ranking  $N$ -tuple  $\mathbf{P}$ ;
- (2) for  $i \in \{1, \dots, N\}$ ,  $f_i$  is  $i$ 's unilateral component of  $d - f_0$ ; and
- (3) for all pairs  $x, z$ ,  $f_{xz}$  is the  $xz$  duple component of  $(d - f_0) - \sum_{i=1}^N f_i$ .

7. PROOF OF THEOREM 1

*Sufficiency.* It is easily checked that point voting decision schemes, supporting size decision schemes and their probability mixtures are anonymous and neutral. To show that they are strategy-proof, let  $d_a$  be any point voting scheme with positional scoring vector  $(a_1, \dots, a_M)$  and  $d_b$  any supporting size decision scheme, with supporting size scoring vector  $(b_N, \dots, b_0)$ , and let

$$d = \alpha d_a + (1 - \alpha) d_b, \quad \text{for } 0 \leq \alpha \leq 1.$$

For each  $i \in I$ , let  $d_{ai}$  be the decision scheme defined by

$$d_{ai}(x, \mathbf{P}) = N a_{r(i,x,\mathbf{P})} \quad \text{for all } x \in V \text{ and all ranking } N\text{-tuples } \mathbf{P}.$$

Clearly, each  $d_{ai}$  is unilateral, localized and non-perverse. For each pair  $x, y \in V$ , let  $d_{bxy}$  be the decision scheme defined by

$$d_{bxy}(z, \mathbf{P}) \begin{cases} 0 & \text{if } z \notin \{x, y\} \\ [M(M-1)/2] b_{s(x,y,\mathbf{P})} & \text{if } z = x \\ [M(M-1)/2] b_{s(y,x,\mathbf{P})} & \text{if } z = y. \end{cases}$$

Clearly, each  $d_{bxy}$  is duple, localized and non-perverse. In addition, it

is easily checked that

$$d_a = \sum_{i=1}^N (1/N)d_{ai} \quad \text{and that} \quad d_b = (2/M(M-1)) \sum_{x \in V} \sum_{y \neq x} d_{bxy},$$

so that  $d$  can be written as

$$d = \sum_{i=1}^N (\alpha/N)d_{ai} + \sum_{x \in V} \sum_{y \neq x} (2(1-\alpha)/M(M-1))d_{bxy}, \quad \text{i.e.,}$$

as a probability mixture of localized and non-perverse decision schemes, each of which is either unilateral or duple. Thus, by Gibbard's Theorem,  $d$  is strategy proof.

*Necessity.* Let  $d$  be strategy-proof, neutral and anonymous. Since  $d$  is strategy-proof, it can be decomposed as in Theorem 2', in the form

$$d = f_0 + \sum_{i=1}^N f_i + \sum_{x \in V} \sum_{\substack{y \neq x \\ z \in V}} f_{xz} = f_0 + f^*.$$

(a)  $f_0(x, \mathbf{P}) = f_0(y, \mathbf{P})$  for all  $x, y$  and  $\mathbf{P}$ .

*Proof.* Suppose not. Then, there would exist two alternatives  $x$  and  $y$  such that, for any ranking  $N$ -tuple  $\mathbf{P}$ ,  $f_0(x, \mathbf{P}) \neq f_0(y, \mathbf{P})$ . Let  $\rho$  be a permutation on  $V$  for which  $y = \rho(x)$ ,  $x = \rho(y)$  and  $z = \rho(z)$  for all  $z \notin \{x, y\}$ . Let  $\hat{\mathbf{P}}$  be a ranking  $N$ -tuple under which  $x$  is the lowest alternative and  $y$  is the alternative before last in every individual's ranking. Then, clearly,  $\Gamma_x \hat{\mathbf{P}} = \hat{\mathbf{P}}$ ,  $\Gamma_y \hat{\mathbf{P}} = \hat{\mathbf{P}}^\rho$ .

By definition of  $f_0$ , we have:

$$f_0(x, \hat{\mathbf{P}}) = d(x, \Gamma_x \hat{\mathbf{P}}) = d(x, \hat{\mathbf{P}})$$

$$f_0(y, \hat{\mathbf{P}}) = d(y, \Gamma_y \hat{\mathbf{P}}) = d(y, \hat{\mathbf{P}}^\rho).$$

By hypothesis,  $f_0(x, \hat{\mathbf{P}}) \neq f_0(y, \hat{\mathbf{P}})$ . Thus,  $d(x, \hat{\mathbf{P}}) \neq d(\rho(x), \hat{\mathbf{P}}^\rho)$ , contradicting the fact that  $d$  is neutral.

Therefore, it must be that  $f_0(x, \mathbf{P}) = f_0(y, \mathbf{P})$  for all  $x, y$  and  $\mathbf{P}$ , as stated. Clearly, this implies that  $f_0$  is neutral and anonymous.

(b)  $f^*$  is neutral and anonymous.

*Proof.* Let, in general,  $f$ ,  $f'$  and  $f''$  be three schemes such that  $f = f' + f''$ . From our definitions, the following two statements must hold:

- If  $f$  and  $f'$  are anonymous,  $f''$  is anonymous.
- If  $f$  and  $f'$  are neutral,  $f''$  is neutral.

Now, in our case,  $d = f_0 + f^*$ .  $d$  is neutral and anonymous by assumption. It has been proven in (a) that  $f_0$  is also neutral and anonymous. Thus,  $f^*$  must be neutral and anonymous.

(c) The unilateral components of  $d$  are identical for all individuals. By this I mean that, if  $\mathbf{P}$  and  $\mathbf{P}'$  are any two ranking  $N$ -tuples such that  $P_i = P'_j$  for some  $i, j \in I$ , then  $f_i(x, \mathbf{P}) = f_j(x, \mathbf{P}')$  for all  $x \in V$ .

*Proof.* By definition, for all  $x$  and  $\mathbf{P}$ ,  $f_k(x, \mathbf{P})$  is the minimal value, from among all  $\mathbf{P}^*$  such that  $P_k = P_k^*$ , of  $f^*(x, \mathbf{P}^*) - f^*(x, \mathbf{P}^*/_k \Gamma_x P_k)$ . Suppose, contrary to the assertion, that there were two individuals whose corresponding unilateral components were not identical. That is, for some  $\mathbf{P}', \mathbf{P}''$  such that  $P'_j = P''_h$  and some  $x \in V$ ,  $f_h(x, \mathbf{P}'') \neq f_j(x, \mathbf{P}')$ .

Suppose, without loss of generality, that  $f_h(x, \mathbf{P}'') < f_j(x, \mathbf{P}')$ , and let  $\bar{\mathbf{P}}$  be the specific ranking  $N$ -tuple among those with  $P_h = P''_h$  for which  $f^*(x, \bar{\mathbf{P}}) - f^*(x, \bar{\mathbf{P}}/_h \Gamma_x P''_h) = f_h(x, \mathbf{P}'')$ .

Let  $\sigma$  be the permutation of individuals that switches  $h$  and  $j$ , keeping the positions of all other individuals. Since  $f^*$  is anonymous by (b),  $f^*(x, \bar{\mathbf{P}}) = f^*(x, \bar{\mathbf{P}}^\sigma)$ , and  $f^*(x, \bar{\mathbf{P}}/_h \Gamma_x P''_h) = f^*(x, [\bar{\mathbf{P}}/_h \Gamma_x P''_h]^\sigma) = f^*(x, \bar{\mathbf{P}}^\sigma/_j \Gamma_x P'_j)$ .

Thus, for  $\bar{\mathbf{P}}^\sigma$  we would have that  $\bar{P}_j = P'_j$ , and that

$$f^*(x, \bar{\mathbf{P}}^\sigma) - f^*(x, \bar{\mathbf{P}}^\sigma/_j \Gamma_x P'_j) < f_j(x, \mathbf{P}'),$$

in contradiction with the definition of  $f_j$ , which requires it to give the minimal value of all such differences.

(d) The value of each of the unilateral components of  $d$  at any given profile for any given alternative, depend only on the rank of this alternative for the corresponding individual at the given profile.

*Proof.* Suppose not. Then there would be  $i, \mathbf{P}, \mathbf{P}'$ ,  $x$  such that  $r(i, x, \mathbf{P}) = r(i, x, \mathbf{P}')$  and yet  $f_i(x, \mathbf{P}) < f_i(x, \mathbf{P}')$ .

Let  $\rho$  be a permutation on the set of alternatives such that, for all  $z$ ,  $r(i, z, \mathbf{P}) = r(i, \rho(z), \mathbf{P}')$ , i.e.,  $P_i = P'_i$ . Thus, in particular,  $\rho(x) = x$ .

Let  $\bar{\mathbf{P}}$  be the specific ranking  $N$ -tuple for which  $P_i = \bar{P}_i$  and  $f_i(x, \mathbf{P}) = f^*(x, \bar{\mathbf{P}}) - f^*(x, \bar{\mathbf{P}}/i \Gamma_x \bar{\mathbf{P}}_i)$ .

Since  $f^*$  is neutral,  $f^*(x, \bar{\mathbf{P}}) = f^*(x, \bar{\mathbf{P}}^\rho)$ , and  $f^*(x, \bar{\mathbf{P}}/i \Gamma_x \bar{\mathbf{P}}_i) = f^*(x, [\bar{\mathbf{P}}/i \Gamma_x \bar{\mathbf{P}}_i]^\rho) = f^*(x, \bar{\mathbf{P}}^\rho/i \Gamma_x \bar{\mathbf{P}}'_i)$ , where  $\bar{\mathbf{P}}'_i = P'_i$ . But then, it would be that  $f_i(x, \bar{\mathbf{P}}) = f^*(x, \bar{\mathbf{P}}^\rho) - f^*(x, \bar{\mathbf{P}}^\rho/i \Gamma_x \bar{\mathbf{P}}'_i) < f_i(x, \mathbf{P}')$ , in contradiction with the definition of  $f_i$ .

(e) Let  $W_0$  be the weight of  $f_0$ . Let  $W(i)$  be the weight of  $f_i$ , and  $W_1 = \sum_{i=1}^N W(i)$ . Then,  $d^s = (1/W_0 + W_1)[f_0 + \sum_{i=1}^N f_i]$  is a point voting decision scheme. It is thus neutral, anonymous, localized and non-perverse.

*Proof.* By (c) and (d),  $f_j(x, \mathbf{P}) = f_j(y, \mathbf{P}')$  for any  $i, j, x, y, \mathbf{P}$  and  $\mathbf{P}'$  such that  $r(j, x, \mathbf{P}) = r(j, y, \mathbf{P}')$ . Let, for each given  $h$ ,  $a'_h = f_i(x, \mathbf{P})(1/W_0 + W_1)$  for any  $x$  and  $\mathbf{P}$  such that  $r(i, x, \mathbf{P}) = h$ .

Since  $d$  is non-perverse and  $f_0$  is constant,  $f^*$  must be non-perverse. Then, by the definition of  $f_i$ , each  $f_i$  must be non-perverse, and this implies that  $a_h \geq a_k$  whenever  $h < k$ .

Let, for each  $1 \leq h \leq M$ ,  $a_h = a'_h + (W_0/NM)(1/W_0 + W_1)$ . It is left to the reader to check that  $d^s = (1/W_0 + W_1)[f_0 + \sum_{i=1}^N f_i]$ .

(f) Let  $d^{ss} = (1/W_2) \sum_{x \in V} \sum_{\substack{y \in V \\ y \neq x}} f_{xy}$ , where  $W(x, y)$  is the weight of  $f_{xy}$ ,

and  $W_2 = \sum_{x \in V} \sum_{\substack{y \in V \\ y \neq x}} W(x, y)$ . Then,  $d^{ss}$  is a supporting size decision scheme.

*Proof.* The proof of (f) will only be outlined here. It involves a line of reasoning similar to that of (b), (c), (d) and (e).

By definition,  $f_0 + \sum_{i=1}^N f_i = (d^s/W_0 + W_1)$ . Thus,  $d = (d^s/W_0 + W_1) + \sum_{x \in V} \sum_{\substack{y \in V \\ x \neq y}} f_{xy}$ . Since  $d^s$  is neutral and anonymous,  $(d^s/W_0 + W_1)$  must also be. This, and our hypothesis on  $d$ , implies that  $\hat{f} = \sum_{x \in V} \sum_{\substack{y \in V \\ w \neq y}} f_{xy}$  must be neutral and anonymous.

By a reasoning similar to the one in (c) and (d), and using the fact that  $f_{xy}(x, \mathbf{P}) = \hat{f}(x, \Gamma_{xy} \mathbf{P})$ , we could show that if  $\mathbf{P}$  and  $\mathbf{P}'$  are two ranking  $N$ -tuples such that, for all  $i \in I$  and any  $x, y, z, w \in V$ ,  $xP_i y \Leftrightarrow zP'_i w$ , then  $f_{xy}(x, \mathbf{P}) = f_{zw}(z, \mathbf{P}')$ . That is,  $f_{xy}(x, \mathbf{P}) = f_{zw}(z, \mathbf{P}')$  for any  $x, y, z, w, \mathbf{P}$  and  $\mathbf{P}'$  such that  $s(x, y, \mathbf{P}) = s(z, w, \mathbf{P}')$ . Let, for each

given  $k$ ,  $0 \leq k \leq M$ ,  $b_k = f_{xy}(x, \mathbf{P})(1/W_2)$  for any  $x, y$  and  $\mathbf{P}$  such that  $s(x, y, \mathbf{P}) = k$ .

Since  $d$  and  $f_0 + \sum_{i=1}^N f_i$  are non-perverse,  $\sum_{x \in V} \sum_{\substack{y \in V \\ x \neq y}} f_{xy}$  must also be non-perverse, and so must be  $f_{xy}$  for any pair  $x, y \in V$ . Therefore,  $b_k \geq b_h$  whenever  $k > h$ .

It is now left to the reader to check that  $d^{ss}$  is indeed a supporting size decision scheme, with supporting size vector  $(b_N, \dots, b_0)$ .

(g) By Gibbard's Theorem,  $d = f_0 + \sum_{i=1}^N f_i + \sum_{x \in V} \sum_{\substack{y \in V \\ x \neq y}} f_{xy}$ . Thus,  $d = (W_0 + W_1)d^s + W_2d^{ss}$ , where  $W_0 + W_1 + W_2 = 1$ ,  $W_0 + W_1 \geq 0$ , and  $W_2 \geq 0$ . Therefore,  $d$  can be expressed as a probability mixture of a point voting decision scheme  $d^s$  and a supporting size decision scheme  $d^{ss}$ , and this completes the proof of the theorem.

## 8. SOME ADDITIONAL REQUIREMENTS

What has been said up to now shows that point voting schemes and supporting size schemes, as well as their probability mixtures, exhibit a number of desirable properties. Some additional features of such schemes are now discussed. Proofs of the assertions are left to the reader.

First of all notice that point voting decision schemes, supporting size decision schemes and their probability mixtures all satisfy the following minimal efficiency property:

*Weak unanimity:* For any  $\mathbf{P}$ ,  $x$  and  $y$ ,

$$(\forall i)xP_iy \rightarrow d(x, \mathbf{P}) \geq d(y, \mathbf{P}).$$

A different efficiency property can also be imposed upon schemes:

*Rejection:* For any  $\mathbf{P}$  and  $z$ ,

$$(\forall y)[(\forall i)yP_i z] \rightarrow d(z, \mathbf{P}) = 0.$$

Clearly, for a neutral, anonymous and strategy-proof decision scheme to satisfy the rejection property it has to be decomposable into a point voting decision scheme which assigns weight zero to the last-ranked alternative and a supporting size decision scheme for which  $b_0 = 0$ .

Finally, consider the following condition, which implies both weak unanimity and rejection:

*Ex-post Pareto Optimality*: For any  $\mathbf{P}$ ,  $x$  and  $y$ ,

$$(\forall i)xP_i y \rightarrow d(y, \mathbf{P}) = 0.$$

The only neutral, anonymous and strategy proof decision scheme which satisfies Ex-post Pareto Optimality is the point voting scheme with positional scoring vector  $(1, 0, 0, \dots, 0)$ .<sup>2</sup>

## 9. CONCLUSIONS

A class of decision schemes which appears to be 'nice' from several points of view has been characterized. In particular, the condition of strategy-proofness, which is known to be a very rare feature among deterministic social decision-making procedures, is shown to be compatible with several other desirable criteria within the probabilistic framework. By operating over a larger domain, decision schemes are able to generate social outcomes which do not leave any room for the type of bargaining 'within the rules' by means of preference misrepresentation to which manipulable procedures are open. The main causes of dissatisfaction with the subclasses of strategy-proof decision schemes which are characterized above are to be found (1) on their poor efficiency properties<sup>3</sup> and/or (2) on the possibility that the very notion of letting chance play an important role might be unacceptable for certain social decision-making problems.<sup>4</sup>

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## NOTES

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<sup>1</sup> This terminology is due to A. Gibbard [4]. Gibbard's notation and definitions are used here whenever possible.

<sup>2</sup> This procedure has been called the 'random dictator' system. See [4], Section 5.

<sup>3</sup> See Section 8 and Gibbard [4].

<sup>4</sup> See [1].

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